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The effect of variable properties on momentum and heat transfer in a tube with constant heat flux across the wall

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Abstract—The influence of variable properties on laminar fully developed pipe flow is analyzed within the framework of an asymptotic theory for small heat transfer rates. The functions describing the temperature dependence of the fluid properties are expanded as Taylor series at the reference state, whose coefficients are not specified for particular fluids. Shear stress and temperature at the wall are given as universal power series of the parameter $\varepsilon = \frac{11}{24} q_W$.

For all fluids it is shown how the exponents in the property-ratio method depend on the fluid properties. Hence, this method is not an empirical one any more.

1. INTRODUCTION

A FULLY developed laminar tube flow is considered which has the well-known parabolic velocity profile if constant properties are assumed. Skin friction and heat transfer are characterized by constant numbers for the friction factor $f \cdot Re$ and the Nusselt number Nu.

But, of course, this is only an approximation since in reality all physical properties involved in the problem are dependent on temperature and pressure. And, indeed, there is a pressure drop in the tube and there are temperature gradients as soon as heat transfer is considered.

For practical purposes, it is desirable to know how the constant-property approximations can be corrected for the influence of variable fluid properties. A well-known method of that kind is the property ratio method. Here the constant property results for $f \cdot Re$ and Nu are multiplied by the ratio of certain properties at different temperatures to the power of some exponent thus taking into account the temperature dependence of the properties.

This method is not rational since the exponents are empirical numbers and the properties involved in the correction factor are more or less chosen intuitively.

In the following a systematic way of correcting the constant property results is presented. It turns out that under certain conditions only the temperature dependence of the properties must be taken into account. By expanding the properties as Taylor series at a reference state one finally comes out with correction factors for $f \cdot Re$ and Nu without a need for any empirical information. It is a rational asymptotic approximation for small heating rates which can be systematically improved by higher order terms.

2. BASIC EQUATIONS

The Navier–Stokes equations for steady flows in cylindrical coordinates, nondimensionalized and transformed according to Table 1 read (viscous heating ($Ma \to 0$, see below) and buoyancy forces are neglected), see for example [1]):

continuity:

$$\frac{\partial}{\partial x}(\rho u) + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v) = 0 \tag{1}$$

x-momentum:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\eta\frac{\partial u}{\partial r}\right) \tag{2}$$

r-momentum:

$$0 = -\frac{\partial p}{\partial r} \tag{3}$$

energy (enthalpy):

$$\rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial r}\right) = \frac{1}{Pr_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\lambda}{c_p}\frac{\partial h}{\partial r}\right), \tag{4}$$

$$Pr_0 = \eta_0^* c_{p0}^* / \lambda_0^*$$

with associated boundary conditions:

$$u = v = 0; \quad \frac{\partial h}{\partial r} = \frac{c_{pW}}{\lambda_W} q_W \quad \text{at} \quad r = 1 \tag{5}$$

$$v = \frac{\partial u}{\partial r} = \frac{\partial h}{\partial r} = 0 \quad \text{at} \quad r = 0. \tag{6}$$

Equations (1)–(4) are a simplified form of the full Navier–Stokes equations valid for cases in which the

NOMENCLATURE

A	cross-sectional area
c_p	specific heat at constant pressure
f	friction factor, equation (53)
$\tilde{f}$	friction factor, equation (66)
h	specific enthalpy
K_α	dimensionless property, equation (11)
$\tilde{K}_\alpha$	dimensionless property, equation (12)
Ma	Mach number
m_i	exponent, equation (72)
$\dot{m}$	mass flux
Nu	Nusselt number, equation (58)
$\tilde{Nu}$	Nusselt number, equation (69)
n_i	exponent, equation (71)
p	pressure
Pr	Prandtl number
q	heat flux
r	radial coordinate
R	pipe radius
Re	Reynolds number
T	temperature
u, v	velocity components
U_0	mean velocity at reference conditions
x	axial coordinate
y	coordinate perpendicular to the wall.

Greek symbols

α	physical property
γ	ratio of heat capacities
ε	perturbation parameter, equation (14)
η	viscosity
λ	thermal conductivity
ρ	density
τ_W	wall shear stress.

Subscripts

B	bulk
c.p.	constant property
0	reference condition
W	wall.

ratio of a characteristic radial dimension (R^* for example) to a characteristic streamwise dimension (L^* for example) is small.

In the limit $R^*/L^* \to 0$ which holds for fully developed pipe flows the Navier–Stokes equations can be drastically simplified as can be seen with equation (3) for example. This limiting process can be performed in such a way (see Table 1) that the Reynolds number is no longer explicitly present in the equations.

All variables are nondimensionalized with quantities at the reference state '0', as explained in Fig. 1. Far away from the inlet of a pipe the flow is hydrodynamically fully developed. This fully developed state is our reference state '0'. At a certain point ($x^* = 0$) in the tube heat transfer across the pipe wall with $q_W = \text{const.}$ will start which leads to a linear increase in bulk enthalpy as shown in Fig. 1.

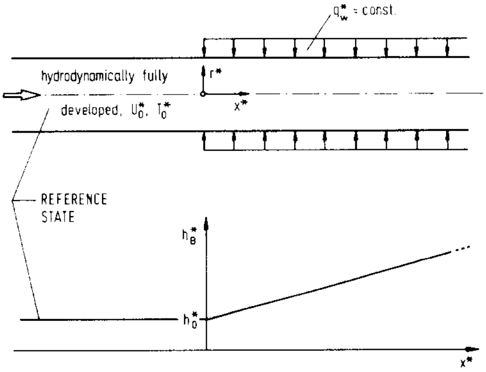

FIG. 1. Reference state; bulk enthalpy distribution.

The physical properties involved in the problem are density ρ, viscosity η, thermal conductivity λ and specific heat capacity c_p. As c_p is not constant the energy equation is written in enthalpy h rather than temperature T. The relation between h and T is:

$$dh = c_p \, dT \quad \text{or} \quad h = \int_0^T c_p \, dT. \tag{7}$$

The Reynolds and Prandtl numbers are defined with properties at the reference state:

$$Re_0 = \frac{\rho_0^* U_0^* R^*}{\eta_0^*}; \quad Pr_0 = \frac{\eta_0^* c_{p0}^*}{\lambda_0^*} \tag{8}$$

The reference velocity $U_0^* = \dot{m}^*/\rho_0^* A^*)$ is the mean velocity of the oncoming fully developed pipe flow.

3. PHYSICAL PROPERTIES

All physical properties involved are more or less pressure and temperature dependent. A systematic way to quantify this statement is to expand these properties

Table 1. Dimensionless, transformed variables ($* \mathrel{\hat{=}}$ dimensional quantities)

x	r	u	v	p	T	h	ρ	η	λ	c_p
$\frac{x^*}{R^*} Re_0^{-1}$	$\frac{r^*}{R^*}$	$\frac{u^*}{U_0^*}$	$\frac{v^*}{U_0^*} Re_0$	$\frac{p^* - p_0^*}{\rho_0^* U_0^{*2}}$	$\frac{T^* - T_0^*}{T_0^*}$	$\frac{h^* - h_0^*}{c_{p0}^* T_0^*}$	$\frac{\rho^*}{\rho_0^*}$,	$\frac{\eta^*}{\eta_0^*}$,	$\frac{\lambda^*}{\lambda_0^*}$,	$\frac{c_p^*}{c_{p0}^*}$

as Taylor series at the reference state. The physical property α (here ρ, η, λ or c_p) then reads:

$$\alpha^* = \alpha_0^* + \left.\frac{\partial \alpha^*}{\partial T^*}\right|_0 (T^* - T_0^*) + \left.\frac{\partial \alpha^*}{\partial p^*}\right|_0 (p^* - p_0^*) + \cdots \tag{9}$$

or in dimensionless form with $\alpha = \alpha^*/\alpha_0^*$:

$$\alpha = 1 + K_\alpha T + \tilde{K}_\alpha p + \cdots \tag{10}$$

with

$$K_\alpha = \left(\frac{T^*}{\alpha^*}\frac{\partial \alpha^*}{\partial T^*}\right)_0 \tag{11}$$

$$\tilde{K}_\alpha = \left(\frac{\rho^* U^{*2}}{\alpha^*}\frac{\partial \alpha^*}{\partial p^*}\right)_0 = \frac{\rho_0^* U_0^{*2}}{p_0^*}\left(\frac{p^*}{\alpha^*}\frac{\partial \alpha^*}{\partial p^*}\right)_0$$

$$= \frac{\rho_0^* U_0^{*2}}{p_0^*}\bar{K}_\alpha. \tag{12}$$

Here K_α and $\tilde{K}_\alpha$ are dimensionless numbers characterising the fluid like the Prandtl number does. By means of K_α and $\tilde{K}_\alpha$ one can easily select the important dependencies. In Table 2 the values for air and for water are listed, showing that the pressure dependence is almost always negligible as $\bar{K}_{1\alpha}$ is extremely small (and so α is nearly constant with respect to p).

Only gases have considerable values of $\bar{K}_\rho$. For gases $(\partial p^*/\partial \rho^*)_{T0} = a_0^{*2}/\gamma$, with a_0^* being the local speed of sound, and $\tilde{K}_\rho$ is (see [2]):

$$\tilde{K}_\rho = \gamma Ma_0^2. \tag{13}$$

We now restrict ourselves to small Mach numbers ($Ma \to 0$, asymptotically) and so are left over with the physical properties being temperature dependent only.

As a consequence of this for small heat transfer rates and small temperature differences the linear approximation to the functions $\alpha(T)$ according to equation (10) may be sufficient. If higher accuracy is desired an extension of what is presented here as linear theory is straightforward, see [3]. Then, in the Taylor series expansion, equation (9), higher order terms, neglected here, have to be taken into account.

4. ASYMPTOTIC EXPANSIONS

As our intention is to improve constant property results, we start with this simple approximation of the real problem. In Fig. 2 the development of the temperature profile with x is shown. Far enough downstream from the starting point of heat transfer, the so-called hydrodynamically and thermally fully developed state is reached. Not only is the velocity profile independent of x but also the temperature profile relative to the local bulk temperature which is linearly increasing with x, of course. As a consequence of this, there is a constant temperature difference $T_W - T_B$ in that region. We now define this temperature difference as our perturbation parameter ε:

$$\varepsilon = T_{W1} - T_{B1}. \tag{14}$$

The index 1 refers to the constant property solution. The perturbation parameter ε is proportional to the heat transfer rate q_W ($q_W = q_W^* R^*/\lambda_0^* T_0^*$) by:

$$\varepsilon = \tfrac{11}{24} q_W. \tag{15}$$

According to this equation $\varepsilon \to 0$ means vanishing heat transfer rates and vanishing temperature differences. As usual in asymptotic theory we therefore define transformed variables such as for example $\bar{T} = T/\varepsilon$ to get quantities of order one before expanding all variables in series of ε valid asymptotically for $\varepsilon \to 0$.

With:

$$\bar{T} = T/\varepsilon, \quad \bar{h} = h/\varepsilon, \quad \bar{q}_W = q_W/\varepsilon = \tfrac{24}{11} \tag{16}$$

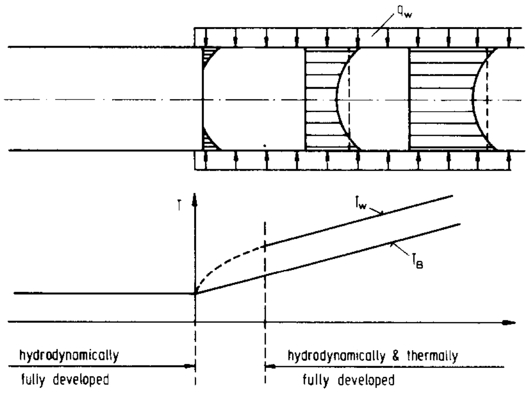

FIG. 2. Development of the temperature profile; constant properties.

Table 2. Variation of physical properties with temperature and pressure for air and water ($T_0 = 293$ K, $p_0 = 1$ bar), from [5] with

$$\bar{K}_\alpha = \left(\frac{p^*}{\alpha^*}\frac{\partial \alpha^*}{\partial p^*}\right)_0.$$

	K_ρ	K_η	K_λ	K_c	$\bar{K}_\rho$	$\bar{K}_\eta$	$\bar{K}_\lambda$	$\bar{K}_c$
Air	-1.01	0.70	0.83	0.01	1.0	8.5×10^{-3}	1.3×10^{-3}	1.8×10^{-3}
Water	-0.06	-7.37	0.75	-0.05	5.4×10^{-5}	1.0×10^{-6}	1.5×10^{-4}	-4.4×10^{-5}

the (Taylor series) expansion of the physical properties reads:

$$\rho = 1+K_\rho\varepsilon\bar{T}_1+\mathrm{O}(\varepsilon^2) \tag{17a}$$

$$\eta = 1+K_\eta\varepsilon\bar{T}_1+\mathrm{O}(\varepsilon^2) \tag{17b}$$

$$\lambda = 1+K_\lambda\varepsilon\bar{T}_1+\mathrm{O}(\varepsilon^2) \tag{17c}$$

$$c_\mathrm{p} = 1+K_\mathrm{c}\varepsilon\bar{T}_1+\mathrm{O}(\varepsilon^2). \tag{17d}$$

According to these expansions we suggest the following formulae for the dependent variables of the problem:

$$u = u_1+\varepsilon(K_\rho u_{2\rho}+K_\eta u_{2\eta})+\mathrm{O}(\varepsilon^2) \tag{18a}$$

$$v = v_1+\varepsilon(K_\rho v_{2\rho}+K_\eta v_{2\eta})+\mathrm{O}(\varepsilon^2) \tag{18b}$$

$$p = p_1+\varepsilon(K_\rho p_{2\rho}+K_\eta p_{2\eta})+\mathrm{O}(\varepsilon^2) \tag{18c}$$

$$\bar{h} = \bar{h}_1+\varepsilon(K_\rho\bar{h}_{2\rho}+K_\eta\bar{h}_{2\eta}+K_\lambda\bar{h}_{2\lambda}+K_\mathrm{c}\bar{h}_{2\mathrm{c}})+\mathrm{O}(\varepsilon^2) \tag{18d}$$

$$\bar{T} = \bar{T}_1+\varepsilon(K_\rho\bar{T}_{2\rho}+K_\eta\bar{T}_{2\eta}+K_\lambda\bar{T}_{2\lambda}+K_\mathrm{c}\bar{T}_{2\mathrm{c}})+\mathrm{O}(\varepsilon^2). \tag{18e}$$

All variables with index 1, like u_1, $v_1,\ldots$, describe the constant property solution, those with index 2, like $u_{2\rho}$, $u_{2\eta}$, $v_{2\rho},\ldots$, together with the constants K_ρ, $K_\eta,\ldots$, are the coefficients of the *linear* (in ε) deviations according to the temperature dependence of the physical properties. That is why the following theory is called *linear theory*, neglecting deviations of order $\mathrm{O}(\varepsilon^2)$. With equations (17d), (18d) and (18e) the relation between temperature and enthalpy, equation (7), is:

$$\bar{h}_1 = \bar{T}_1;\quad \bar{h}_{2\rho} = \bar{T}_{2\rho};\quad \bar{h}_{2\eta} = \bar{T}_{2\eta};\quad \bar{h}_{2\lambda} = \bar{T}_{2\lambda}; \tag{19a}$$

$$\bar{h}_{2\mathrm{c}} = \bar{T}_{2\mathrm{c}}+\tfrac{1}{2}\bar{T}_1^2. \tag{19b}$$

Equations (17) and (18) inserted into the basic equations (1)–(4) lead to the following system of partial differential equations when terms of equal magnitude in $(\varepsilon K_\alpha)^n$ with $n = 0$; 1 and $\alpha = \rho$, η, λ, c_p are collected.

First order system
($n = 0$, constant properties):

$$\frac{\partial u_1}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv_1) = 0 \tag{20}$$

$$u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial r} = -\frac{\mathrm{d}p_1}{\mathrm{d}x}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right) \tag{21}$$

$$u_1\frac{\partial\bar{h}_1}{\partial x}+v_1\frac{\partial\bar{h}_1}{\partial r} = \frac{1}{Pr_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{h}_1}{\partial r}\right) \tag{22}$$

with the associated boundary conditions:

$$r = 1:\quad u_1 = v_1 = 0,\quad \frac{\partial\bar{h}_1}{\partial r} = \bar{q}_\mathrm{w} \tag{23a}$$

$$r = 0:\quad v_1 = 0,\quad \frac{\partial u_1}{\partial r} = \frac{\partial\bar{h}_1}{\partial r} = 0. \tag{23b}$$

Second order system
($n = 1$)

continuity:

$$\frac{\partial u_{2\rho}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv_{2\rho}) = -\left[\frac{\partial}{\partial x}(\bar{T}_1u_1)+\frac{1}{r}\frac{\partial}{\partial r}(r\bar{T}_1v_1)\right] \tag{24}$$

$$\frac{\partial u_{2\eta}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rv_{2\eta}) = 0 \tag{25}$$

momentum:

$$u_{2\rho}\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_{2\rho}}{\partial x}+v_{2\rho}\frac{\partial u_1}{\partial r}+v_1\frac{\partial u_{2\rho}}{\partial r} = -\frac{\mathrm{d}p_{2\rho}}{\mathrm{d}x}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{2\rho}}{\partial r}\right)-\bar{T}_1\left(u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial r}\right) \tag{26}$$

$$u_{2\eta}\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_{2\eta}}{\partial x}+v_{2\eta}\frac{\partial u_1}{\partial r}+v_1\frac{\partial u_{2\eta}}{\partial r} = -\frac{\mathrm{d}p_{2\eta}}{\mathrm{d}x}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_{2\eta}}{\partial r}+\bar{T}_1\frac{\partial u_1}{\partial r}\right)\right] \tag{27}$$

energy:

$$u_{2\rho}\frac{\partial\bar{h}_1}{\partial x}+u_1\frac{\partial\bar{h}_{2\rho}}{\partial x}+v_{2\rho}\frac{\partial\bar{h}_1}{\partial r}+v_1\frac{\partial\bar{h}_{2\rho}}{\partial r} = \frac{1}{Pr_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{h}_{2\rho}}{\partial r}\right)-\bar{h}_1\left(u_1\frac{\partial\bar{h}_1}{\partial x}+v_1\frac{\partial\bar{h}_1}{\partial r}\right) \tag{28}$$

$$u_{2\eta}\frac{\partial\bar{h}_1}{\partial x}+u_1\frac{\partial\bar{h}_{2\eta}}{\partial x}+v_{2\eta}\frac{\partial\bar{h}_1}{\partial r}+v_1\frac{\partial\bar{h}_{2\eta}}{\partial r} = \frac{1}{Pr_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{h}_{2\eta}}{\partial r}\right) \tag{29}$$

$$u_1\frac{\partial\bar{h}_{2\lambda}}{\partial x}+v_1\frac{\partial\bar{h}_{2\lambda}}{\partial r} = \frac{1}{Pr_0}\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial\bar{h}_{2\lambda}}{\partial r}+\bar{h}_1\frac{\partial\bar{h}_1}{\partial r}\right)\right] \tag{30}$$

$$u_1\frac{\partial\bar{h}_{2\mathrm{c}}}{\partial x}+v_1\frac{\partial\bar{h}_{2\mathrm{c}}}{\partial r} = \frac{1}{Pr_0}\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial\bar{h}_{2\mathrm{c}}}{\partial r}-\bar{h}_1\frac{\partial\bar{h}_1}{\partial r}\right)\right]. \tag{31}$$

From equation (3) one gets $\partial p_{2\rho}/\partial r = \partial p_{2\eta}/\partial r = 0$ which is taken into account by the *ordinary differentiation* of the pressure in equations (26) and (27).

The associated boundary conditions following from equations (5) and (6) are:

$$r = 1:\quad u_{2\rho} = u_{2\eta} = v_{2\rho} = v_{2\eta} = 0$$

$$\frac{\partial\bar{h}_{2\rho}}{\partial r} = \frac{\partial\bar{h}_{2\eta}}{\partial r} = 0,\quad \frac{\partial\bar{h}_{2\lambda}}{\partial r} = -\frac{\partial\bar{h}_{2\mathrm{c}}}{\partial r} = -\frac{24}{11}\bar{T}_{1\mathrm{w}} \tag{32a}$$

$$r = 0:\quad v_{2\rho} = v_{2\eta} = 0;$$

$$\frac{\partial\bar{h}_{2\rho}}{\partial r} = \frac{\partial\bar{h}_{2\eta}}{\partial r} = \frac{\partial\bar{h}_{2\lambda}}{\partial r} = \frac{\partial\bar{h}_{2\mathrm{c}}}{\partial r} = 0. \tag{32b}$$

In addition to these boundary conditions a number of integral conditions hold. As the mass flow rate, $\dot{m}$, must be the same for constant and variable properties from

$$2\int_0^1 \rho u r \, dr = 1 \quad (\dot{m} = \text{const.}) \tag{33}$$

with (17) and (18) follows:

$$\int_0^1 (\bar{T}_1 u_1 + u_{2\rho}) r \, dr = 0 \tag{34}$$

$$\int_0^1 u_{2\eta} r \, dr = 0. \tag{35}$$

The bulk enthalpy $\bar{h}_B$ in dimensionless variables for $\bar{q}_W = \text{const.}$ is:

$$\bar{h}_B \equiv 2\int_0^1 \rho u \bar{h} r \, dr = \frac{2\bar{q}_W}{Pr_0} x. \tag{36}$$

With equations (17) and (18) inserted into (36) one gets:

$$\int_0^1 (u_1 \bar{h}_1^2 + \bar{h}_1 u_{2\rho} + u_1 \bar{h}_{2\rho}) r \, dr = 0 \tag{37}$$

$$\int_0^1 (\bar{h}_1 u_{2\eta} + u_1 \bar{h}_{2\eta}) r \, dr = 0 \tag{38}$$

$$\int_0^1 u_1 \bar{h}_{2\lambda} r \, dr = 0 \tag{39}$$

$$\int_0^1 u_1 \bar{h}_{2c} r \, dr = 0. \tag{40}$$

5. SOLUTIONS

First order equations, equations (20)–(23):

The first order solutions (constant properties) are well known, in our dimensionless variables they read:

$$u_1 = 2(1 - r^2); \quad v_1 = 0; \quad \frac{dp_1}{dx} = -8 \tag{41}$$

$$\bar{T}_1 = \bar{T}_{11}(x) + \bar{T}_{12}(r), \quad \bar{T}_{11} = \frac{48}{11 Pr_0} x,$$

$$\bar{T}_{12} = \tfrac{6}{11}(-r^4 + 4r^2 - \tfrac{7}{6}) \tag{42}$$

Second order equations, equations (24)–(32), (34), (35), (37)–(40):

The partial differential equations can be integrated with the following formulae:

$$\left.\begin{aligned} & u_{2i} = x u_{2i1}(r) + u_{2i2}(r) && \\ & v_{2i} = 0 && i = \rho, \eta \\ & p_{2i} = a_{1i} x^2 + a_{2i} x && j = \rho, \eta, \lambda, c \\ & \bar{h}_{2j} = x \bar{h}_{2j1}(r) + \bar{h}_{2j2}(r). && \end{aligned}\right\} \tag{43}$$

The solutions are:

$$u_{2\rho} = -\frac{96}{11 Pr_0}\left[(1 - r^2)x + \frac{r^6}{18} - \frac{r^4}{4} + \frac{r^2}{4} - \frac{1}{18}\right] \tag{44}$$

$$u_{2\eta} = \tfrac{2}{11}(-2r^6 + 12r^4 - 13r^2 + 3) \tag{45}$$

$$v_{2\rho} = v_{2\eta} = 0 \tag{46}$$

$$p_{2\rho} = \frac{96}{11 Pr_0}(2x^2 + x) \tag{47}$$

$$p_{2\eta} = -\frac{48}{11}\left(\frac{4}{Pr_0} x^2 + x\right) \tag{48}$$

$$\begin{aligned} \bar{h}_{2\rho} = \frac{1}{121}\Bigg[& \left(9 - \frac{4}{Pr_0}\right) r^8 - \left(80 - \frac{32}{Pr_0}\right) r^6 \\ & + \left(186 - \frac{72}{Pr_0}\right) r^4 - \left(168 - \frac{64}{Pr_0}\right) r^2 \\ & + \left(\frac{59}{5} - \frac{68}{15 Pr_0}\right)\Bigg] \end{aligned} \tag{49}$$

$$\bar{h}_{2\eta} = \tfrac{1}{121}(-3r^8 + 32r^6 - 78r^4 + 72r^2 - 10) \tag{50}$$

$$\begin{aligned} \bar{h}_{2\lambda} = \frac{1}{121}\Bigg[& \frac{48}{Pr_0}(6r^4 - 24r^2 + 7)x - 27r^8 \\ & + 224r^6 - 516r^4 + 336r^2 - \frac{233}{5}\Bigg] \end{aligned} \tag{51}$$

$$\bar{h}_{2c} = -\bar{h}_{2\lambda}. \tag{52}$$

6. MOMENTUM TRANSFER

With the dimensionless wall shear stress $\tau_W = \tau_W^*/(\rho_0^* U_0^{*2})$ we define a friction factor f:

$$f \equiv \frac{\tau_W^*}{\frac{\rho_0^*}{2} U_0^{*2}} = 2\tau_W. \tag{53}$$

From conservation of momentum we get, see [4]:

$$f\, Re_0 = -2\frac{d}{dx}\int_0^1 (p + \rho u^2) r \, dr. \tag{54}$$

In the case of constant properties equation (54) together with equation (41) gives:

$$f_{\text{c.p.}}\, Re_0 = -2\frac{dp_1}{dx}\int_0^1 r \, dr = 8. \tag{55}$$

The subscript at f means 'constant property at the reference temperature level T_0'.

With p, ρ, u according to equations (18c), (17a) and (18a) respectively, from equations (54) and (55) we get:

$$\begin{aligned} \frac{f}{f_{\text{c.p.}}} = 1 + \varepsilon\Bigg[& (-K_\rho + K_\eta)\bar{h}_B \\ & + \left(-K_\rho \frac{4}{11 Pr_0} + K_\eta \frac{6}{11}\right)\Bigg] + O(\varepsilon^2). \end{aligned} \tag{56}$$

With help of equation (56) $f_{\text{c.p.}}$ can be corrected for the influence of temperature dependent properties. The shortcoming of this equation is the fact that one has to go back to the reference state T_0 to correct the friction factor f at some position x in the region of thermally and hydrodynamically fully developed flow, see Fig. 2. How to overcome this will be shown in Section 8.

It should be mentioned that there is an alternative way to obtain equation (56):

With $\tau_W^* = \eta_W^*(\partial u^*/\partial y^*)_W = -\eta_W^*(\partial u^*/\partial r^*)_W$, and in dimensionless form $\tau_W\, Re_0 = -\eta_W(\partial u/\partial r)_W$ we have for constant properties: $\tau_{W\,\mathrm{c.p.}}\, Re_0 = -(\partial u_1/\partial r)_W = 4$ [see equation (41)], so that:

$$\frac{f}{f_{\mathrm{c.p.}}} = \frac{\tau_W}{\tau_{W\,\mathrm{c.p.}}} = -\tfrac{1}{4}\eta_W\left(\frac{\partial u}{\partial r}\right)_W. \qquad (57)$$

If η and u according to equations (17b) and (18a) are now inserted, one comes out with equation (56).

7. HEAT TRANSFER

The Nusselt number in dimensionless variables is:

$$Nu \equiv \frac{q_W^* 2R^*}{\lambda_0^*(T_W^* - T_B^*)} = \frac{2\bar{q}_W}{\bar{T}_W - \bar{T}_B}. \qquad (58)$$

For constant properties $\bar{T}_W - \bar{T}_B = (T_{W1} - T_{B1})/\varepsilon = 1$ [see equation (14)] so that $Nu_{\mathrm{c.p.}} = 48/11 = 4.36$ [see equation (16)] and

$$\frac{Nu}{Nu_{\mathrm{c.p.}}} = (\bar{T}_W - \bar{T}_B)^{-1}. \qquad (59)$$

The energy equation was solved in terms of enthalpy. Within the linear theory enthalpy h and temperature T are related by

$$T = \frac{\sqrt{1+2K_c h} - 1}{K_c} = h - \tfrac{1}{2}K_c h^2 + \cdots \qquad (60)$$

Equation (60) together with $\bar{T} = T/\varepsilon$ and $\bar{h} = h/\varepsilon$ gives:

$$\bar{T}_W - \bar{T}_B = \bar{h}_W - \bar{h}_B - \tfrac{1}{2}\varepsilon K_c(\bar{h}_{W1}^2 - \bar{h}_B^2) + 0(\varepsilon^2). \quad (61)$$

If $\bar{h}_W$ is now calculated from equations (49)–(52) with $\bar{h}_B$ according to equation (36), one finally comes out with the following equation:

$$\frac{Nu}{Nu_{\mathrm{c.p.}}} = 1 + \varepsilon\left[K_\lambda \bar{h}_B + K_\rho\left(\frac{206}{605} - \frac{232}{1815}\frac{1}{Pr_0}\right) - K_\eta\frac{13}{121} + K_\lambda\frac{148}{605} + K_c\frac{309}{1210}\right] + 0(\varepsilon^2). \quad (62)$$

8. PROPERTY RATIO METHOD

By means of equations (56) and (62) the results for momentum and heat transfer calculated under the assumption of constant properties (at temperature T_0) can be corrected at any position x far enough downstream for any thermally fully developed pipe flow. One should keep in mind that the concept of thermally fully developed flow makes sense only if constant properties are assumed. To improve these results by equations (56) and (62), the fluid properties K_α at reference temperature and the bulk enthalpy [see equation (36)] must be known.

With the concept of what we call 'quasi-constant' properties one can transform equations (56) and (62) to a form known as property ratio formulation with the advantage that there is no need to have recourse to properties at reference conditions T_0. There is only need for properties at local wall and bulk temperatures instead.

Flows with quasi-constant properties are defined as following:

(1) all physical properties are taken at the local (x-dependent) bulk temperature;
(2) the constant property laws hold locally, i.e.

$$8 = \frac{2\eta_0^*(\partial u^*/\partial y^*)_{W0}}{\rho_0^* U_0^{*2}}\,\frac{\rho_0^* U_0^* R^*}{\eta_0^*} = \frac{2\eta_B^*(\partial u^*/\partial y^*)_{WB}}{\rho_B^* U_B^{*2}}\,\frac{\rho_B^* U_B^* R^*}{\eta_B^*} \qquad (63)$$

$$4,36 = \frac{q_W^* 2R^*}{\lambda_0^*(T_W^* - T_B^*)_0} = \frac{q_W^* 2R^*}{\lambda_B^*(T_W^* - T_B^*)_B}. \qquad (64)$$

If we now write

$$f_{\mathrm{B.c.p.}} = \frac{2\eta_B^*(\partial u^*/\partial y^*)_{WB}}{\rho_0^* U_0^{*2}}$$

according to the definition of f, equation (53), we get:

$$\frac{f_{\mathrm{B.c.p.}}}{f_{\mathrm{c.p.}}} = \frac{\eta_B^*(\partial u^*/\partial y^*)_{WB}}{\eta_0^*(\partial u^*/\partial y^*)_{W0}} = \frac{\eta_B}{\rho_B} = 1 + \varepsilon(-K_\rho + K_\eta)\bar{h}_B + 0(\varepsilon^2) \quad (65)$$

with $(\partial u^*/\partial y^*)_{WB}/(\partial u^*/\partial y^*)_{W0} = \rho_B^{-1}$ from equation (63).

With the definition:

$$\bar{f} \equiv \frac{\tau_W^*}{\frac{\rho_B^*}{2}U_B^{*2}} = f\rho_B \qquad (66)$$

we can write, combining equations (56) and (65):

$$\frac{f}{f_{\mathrm{B\,c.p.}}} = \frac{\bar{f}}{\bar{f}_{\mathrm{c.p.}}} = \frac{f}{f_{\mathrm{c.p.}}}\,\frac{f_{\mathrm{c.p.}}}{f_{\mathrm{B\,c.p.}}} = 1 + \varepsilon\left(-K_\rho\frac{4}{11 Pr_0} + K_\eta\frac{6}{11}\right) + 0(\varepsilon^2). \quad (67)$$

As $Pr_0 = Pr_B + 0(\varepsilon)$ we finally come out with an equation of local properties, $\bar{f}_{\mathrm{c.p.}}$ according to equation (63) being $\bar{f}_{\mathrm{c.p.}} = 8/Re_B$:

$$\frac{\bar{f}}{\bar{f}_{\mathrm{c.p.}}} = 1 + \varepsilon\left(-K_\rho\frac{4}{11 Pr_B} + K_\eta\frac{6}{11}\right) + 0(\varepsilon^2). \quad (68)$$

In a similar way we define a Nusselt number $\tilde{N}u$:

$$\tilde{N}u \equiv \frac{q_W^* 2R^*}{\lambda_B^*(T_W^* - T_B^*)} = Nu\lambda_B^{-1}. \qquad (69)$$

With $Nu_{\mathrm{c.p.}} = \tilde{N}u_{\mathrm{c.p.}}$ according to equation (64) from equation (62) follows [again we set $Pr_0 = Pr_B + 0(\varepsilon)$]:

$$\frac{\tilde{N}u}{\tilde{N}u_{\mathrm{c.p.}}} = 1 + \varepsilon\left[K_\rho\left(\frac{206}{605} - \frac{232}{1815}\frac{1}{Pr_B}\right) - K_\eta\frac{13}{121} + K_\lambda\frac{148}{605} + K_c\frac{309}{1210}\right] + 0(\varepsilon^2) \quad (70)$$

with $\tilde{N}u_{\mathrm{c.p.}} = 4.36$.

On the basis of equations (68) and (70) we can make the property ratio formulae:

$$\frac{\bar{f}}{\bar{f}_{\mathrm{c.p.}}} = \left[\frac{\rho^*_{\mathrm{W}}}{\rho^*_{\mathrm{B}}}\right]^{n_1}\left[\frac{\eta^*_{\mathrm{W}}}{\eta^*_{\mathrm{B}}}\right]^{n_2} \qquad (71)$$

$$\frac{\tilde{N}u}{\tilde{N}u_{\mathrm{c.p.}}} = \left[\frac{\rho^*_{\mathrm{W}}}{\rho^*_{\mathrm{B}}}\right]^{m_1}\left[\frac{\eta^*_{\mathrm{W}}}{\eta^*_{\mathrm{B}}}\right]^{m_2}\left[\frac{\lambda^*_{\mathrm{W}}}{\lambda^*_{\mathrm{B}}}\right]^{m_3}\left[\frac{c^*_{\mathrm{pW}}}{c^*_{\mathrm{pB}}}\right]^{m_4}. \qquad (72)$$

The property ratios $\alpha^*_{\mathrm{W}}/\alpha^*_{\mathrm{B}}$ with $\alpha = \rho, \eta, \lambda, c_{\mathrm{p}}$ are:

$$\frac{\alpha^*_{\mathrm{W}}}{\alpha^*_{\mathrm{B}}} = \frac{\alpha_{\mathrm{W}}}{\alpha_{\mathrm{B}}} = \frac{1+\varepsilon K_\alpha \bar{T}_{\mathrm{W}} + 0(\varepsilon^2)}{1+\varepsilon K_\alpha \bar{T}_{\mathrm{B}} + 0(\varepsilon^2)}$$

$$= 1+\varepsilon K_\alpha(\bar{T}_{\mathrm{W}} - \bar{T}_{\mathrm{B}}) + 0(\varepsilon^2)$$

and with $\bar{T}_{\mathrm{W}} - \bar{T}_{\mathrm{B}} = 1 + 0(\varepsilon)$, see equations (14), (16) we get:

$$\left[\frac{\alpha^*_{\mathrm{W}}}{\alpha^*_{\mathrm{B}}}\right]^{\beta} = (1+\varepsilon K_\alpha + 0(\varepsilon^2))^{\beta} = 1 + \beta\varepsilon K_\alpha + 0(\varepsilon^2). \qquad (73)$$

With equation (73) applied to equations (71) and (72) the exponents n_i, m_i can be determined by comparing equations (71), (68) and equations (72), (70) respectively, see Table 3. The final result with the exponents correct to three places is:

$$\frac{\bar{f}}{\bar{f}_{\mathrm{c.p.}}} = \left[\frac{\rho^*_{\mathrm{W}}}{\rho^*_{\mathrm{B}}}\right]^{-\frac{0.364}{Pr_{\mathrm{B}}}}\left[\frac{\eta^*_{\mathrm{W}}}{\eta^*_{\mathrm{B}}}\right]^{0.545} \qquad (74)$$

$$\frac{\tilde{N}u}{\tilde{N}u_{\mathrm{c.p.}}} = \left[\frac{\rho^*_{\mathrm{W}}}{\rho^*_{\mathrm{B}}}\right]^{0.340-\frac{0.128}{Pr_{\mathrm{B}}}}\left[\frac{\eta^*_{\mathrm{W}}}{\eta^*_{\mathrm{B}}}\right]^{-0.107}$$

$$\times\left[\frac{\lambda^*_{\mathrm{W}}}{\lambda^*_{\mathrm{B}}}\right]^{0.245}\left[\frac{c^*_{\mathrm{pW}}}{c^*_{\mathrm{pB}}}\right]^{0.255}. \qquad (75)$$

9. DISCUSSION

There are five important features of our systematic approach to the variable property flow:

(1) The final result is universal in the sense that it holds for *all* fluids, for example liquid metals, liquids, gases and oils. There is no need to distinguish between liquids and gases as is usually done when the property ratio method is applied, see for example [6].

(2) All physical information is extracted from the basic equations. There is no need to have recourse to empirical information as was necessary in the past. So the property ratio method is no longer an empirical method.

(3) As long as the property ratio method is introduced as empirically determined correction to the constant property results the choice of properties is arbitrary to some extent. Often the ratio of viscosities is chosen, sometimes that of Prandtl numbers and in other cases assuming power laws for all physical properties even the ratio of temperatures is chosen instead. (The method is then called 'temperature ratio method', see for example [6].)

Our theory gives the one and only choice of property ratios for a complete correction of the constant property results by the method of property ratios.

(4) We can easily show that it makes no sense to distinguish between 'cooling' and 'heating', as is often found in literature, see [6]. As long as constant exponents are assumed, it is a characteristic feature of this formulae that higher order effects cannot be taken into account. Such higher order effects are based on the nonlinear temperature dependence of the physical properties. If they should be taken into account, the property ratio formulae is no longer a proper one, since then for example m_i and n_i in equations (71) and (72) must be functions of ε. These functions $m_i(\varepsilon)$, $n_i(\varepsilon)$ are linear functions as long as the (nonlinear) temperature dependence is quadatric. Different values of m_i, n_i for cooling and heating respectively are an (obviously bad!) approximation to this linear function. If higher order effects are to be taken into account, one should go back to equations (68) and (70) which in that case include terms of order $0(\varepsilon^2)$. For details see [3].

(5) In the final equations the influence of the physical properties can be checked separately. For fluids with one or more properties being constant the corresponding factor $(\alpha^*_{\mathrm{W}}/\alpha^*_{\mathrm{B}})^{m_i,n_i}$ is 1. Thus, for example, for liquids density is usually taken to be constant, so that the first term of equations (71), (72) is 1, provided m_1 and n_1 are of the order 0(1). This condition is violated if $Pr_{\mathrm{B}} \to 0$, as can be seen from Table 3 for example. Liquid metals have extremely small Prandtl numbers, so that the assumption of constant density must be checked very carefully.

If, for example, we take liquid sodium at 200°C with the physical properties: $Pr_{\mathrm{B}} = 0.0072$, $K_\rho = -0.124$, $K_\eta = -0.918$, equation (68) for example is:

$$\frac{\bar{f}}{\bar{f}_{\mathrm{c.p.}}} = 1+\varepsilon(6.26+0.5)+0(\varepsilon^2)$$

with the factor 6.26 stemming from the density

Table 3. Exponents of equations (71), (72)

n_1	n_2	m_1	m_2	m_3	m_4
$-\frac{4}{11Pr_{\mathrm{B}}}$	$\frac{6}{11}$	$\frac{206}{605}-\frac{232}{1815Pr_{\mathrm{B}}}$	$-\frac{13}{121}$	$\frac{148}{605}$	$\frac{309}{1210}$

dependence and thus dominating the viscosity influence given by the second term 0.5.

We finally want to show that our results compare quite well to those extracted from either experimental data or numerical solutions involving temperature dependent properties. Kays and Perkins [6] give a comprehensive review of laminar and turbulent duct flows. They distinguish between liquids and gases and in some cases between heating and cooling. Neither are necessary in our theory as was explained above.

Liquids

The property ratio result in [6] reads:

$$\frac{\tilde{f}}{\tilde{f}_{\text{c.p.}}} = \left[\frac{\eta_W^*}{\eta_B^*}\right]^n; \quad \begin{matrix} n = 0.50 \text{ (cooling)} \\ n = 0.58 \text{ (heating)} \end{matrix} \tag{76}$$

$$\frac{\widetilde{Nu}}{\widetilde{Nu}_{\text{c.p}}} = \left[\frac{\eta_W^*}{\eta_B^*}\right]^m; \quad m = -0.14 \quad \text{or} \quad -0.11. \tag{77}$$

The formulae shows that viscosity is assumed to have the dominating influence. In terms of our theory, that is $\rho^* = \text{const.}$, $\lambda^* = \text{const.}$, and $c_p^* = \text{const.}$, so that equations (74) and (75) reduce to:

$$\frac{\tilde{f}}{\tilde{f}_{\text{c.p.}}} = \left[\frac{\eta_W^*}{\eta_B^*}\right]^{0.545}; \quad \frac{\widetilde{Nu}}{\widetilde{Nu}_{\text{c.p.}}} = \left[\frac{\eta_W^*}{\eta_B^*}\right]^{-0.107}.$$

The exponent for the skin friction lies exactly between those in [6] for heating and cooling.

For the exponent in equation (77) two different numbers are given stemming from two different sources. The exponent $m = -0.14$ is taken from calculations by Deissler [7]. These calculations are carried out for a fluid with a viscosity law which corresponds approximately to the behavior of liquid metals. Above we have shown how density variations influence skin friction for liquid metals. It might be too crude an approximation to cover all effects with one factor containing viscosity only.

The exponent $m = -0.11$ taken from a study by Yang [8] (see [6]) agrees closely with our exponent which is $m = -0.11$ too, if it is rounded to two places.

Gases

For gases in [6] it is assumed that all physical properties exhibit a power law dependence on temperature, so that the ratio of temperatures is appropriate. The temperature ratio results from [6] are:

$$\frac{\tilde{f}}{\tilde{f}_{\text{c.p.}}} = \left[\frac{T_W^*}{T_B^*}\right]^n \quad \begin{matrix} n = 0.81 \text{ (cooling)} \\ n = 1.00 \text{ (heating)} \end{matrix} \tag{78}$$

$$\frac{\widetilde{Nu}}{\widetilde{Nu}_{\text{c.p.}}} = \left[\frac{T_W^*}{T_B^*}\right]^m \quad m = 0.0. \tag{79}$$

If we assume a power law dependence in our results too, for example that for air at 20°C, written as

$$\frac{\alpha_W^*}{\alpha_B^*} = \left[\frac{T_W^*}{T_B^*}\right]^{\omega_\alpha}; \quad (\omega_\rho ; \omega_\eta ; \omega_\lambda ; \omega_c)$$
$$= (-1 ; 0.7 ; 0.85 ; 0.19) \tag{80}$$

we get:

$$\frac{\tilde{f}}{\tilde{f}_{\text{c.p.}}} = \left[\frac{T_W^*}{T_B^*}\right]^{0.89}; \quad \frac{\widetilde{Nu}}{\widetilde{Nu}_{\text{c.p.}}} = \left[\frac{T_W^*}{T_B^*}\right]^{0.02}$$

which agrees closely with equations (78) and (79). Again the exponent for the skin friction lies between those for cooling and heating.

We have applied an asymptotic theory for small heat transfer rates that is valid for $\varepsilon \to 0$. In practice this theory gives reasonable results for considerable heat transfer rates. The range of applicability of equations (78), (79) in [6] is given by: $0.5 < T_W^*/T_B^* < 3$ for example, which corresponds to a range $-0.5 < \varepsilon(T_0^*/T_B^*) < 2$. If the ratio T_0^*/T_B^* is of order 0(1), $-0.5 < \varepsilon < 2$ shows the wide range that ε can cover still giving reasonable results.

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EFFET DES PROPRIETES VARIABLES SUR LE TRANSFERT DE QUANTITE DE MOUVEMENT ET DE CHALEUR DANS UN TUBE AVEC FLUX DE CHALEUR CONSTANT A LA PAROI

Résumé—L'influence des propriétés variables sur l'écoulement laminaire pleinement établi dans un tube est analysée dans le cadre d'une théorie asymptotique pour des faibles flux thermiques. Les fonctions décrivant la dépendance des propriétés du fluide vis-à-vis de la température sont développées en série de Taylor à partir de l'état de référence, avec des coefficients non spécifiés pour des fluides particuliers. Les contraintes de cisaillement et les températures sur la paroi sont données selon une série puissance du paramètre $\varepsilon = \frac{11}{24} q_w$. Pour tous les fluides, on montre comment les exposants dans la méthode de rapport de propriétés, dépendent des propriétés du fluide. Cette méthode n'est pas du tout empirique.

DER EINFLUSS VARIABLER STOFFWERTE AUF IMPULS- UND WÄRMEÜBERGANG IN ROHREN MIT KONSTANTEM WÄRMESTROM ÜBER DIE WAND

Zusammenfassung—Der Einfluß variabler Stoffwerte auf laminare voll ausgebildete Rohrströmung wird im Rahmen einer asymptotischen Theorie für kleine Wärmeübertragungsraten untersucht. Die Funktionen, welche die Temperaturabhängigkeit der Stoffwerte beschreiben, werden als Taylor-Reihen im Bezugspunkt entwickelt. Die entsprechenden Koeffizienten werden aber nicht für bestimmte Stoffe spezifiziert. Wandschubspannung und Wandtemperatur werden als universelle Potenzgesetze des Parameters $\varepsilon = \frac{11}{24} q_W$ angegeben. Für alle Stoffe wird gezeigt, wie die Exponenten in der Stoffwertverhältnis-Methode von den Stoffwerten abhängen. Damit ist diese Methode keine empirische Methode mehr.

ВЛИЯНИЕ ПЕРЕМЕННЫХ СВОЙСТВ ЖИДКОСТИ НА ПЕРЕНОС ИМПУЛЬСА И ТЕПЛА В ТРУБАХ С ПОСТОЯННОЙ ПЛОТНОСТЬЮ ТЕПЛОВОГО ПОТОКА НА СТЕНКЕ

Аннотация—В рамках асимптотической теории проведен анализ влияния переменных свойств жидкости на ламинарное полностью развитое течение в трубе при малой интенсивности теплопереноса. Функции, описывающие температурную зависимость свойств жидкости, разлагаются в ряды Тэйлора вблизи отсчетного состояния, коэффициенты которых не задаются для конкретных жидкостей. Напряжение сдвига и температура на стенке задаются в виде универсального степенного ряда параметра $\varepsilon = \frac{11}{24} q_w$. Для всех жидкостей показано, каким образом показатели степени в используемом методе зависят от свойств жидкости, откуда следует, что метод больше не является эмпирическим.